Capacitated Warehouse Location Model with Risk Pooling

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Abstract: In this article, we introduce the capacitated warehouse location model with risk pooling (CLMRP), which captures the interdependence between capacity issues and the inventory management at the warehouses. The CLMRP models a logistics system in which a single plant ships one type of product to a set of retailers, each with an uncertain demand. Warehouses serve as the direct intermediary between the plant and the retailers for the shipment of the product and also retain safety stock to provide appropriate service levels to the retailers. The CLMRP minimizes the sum of the fixed facility location, transportation, and inventory carrying costs. The model simultaneously determines warehouse locations, shipment sizes from the plant to the warehouses, the working inventory, and safety stock levels at the warehouses and the assignment of retailers to the warehouses. The costs at each warehouse exhibit initially economies of scale and then an exponential increase due to the capacity limitations. We show that this problem can be formulated as a nonlinear integer program in which the objective function is neither concave nor convex. A Lagrangian relaxation solution algorithm is proposed. The Lagrangian subproblem is also a nonlinear integer program. An efficient algorithm is developed for the linear relaxation of this subproblem. The Lagrangian relaxation algorithm provides near-optimal solutions with reasonable computational requirements for large problem instances. Naval Research Logistics 55: 295–312, 2008

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1. INTRODUCTION

Logistics management is defined by the council of supply chain management professionals as "that part of supply chain management that plans, implements, and controls the efficient, effective forward and reverse flow and storage of goods, services and related information between the point of origin and the point of consumption in order to meet customers' requirements." Traditionally, companies have managed the distribution of goods and storage disparately within different functional departments. However, managers today are well aware that optimizing the logistics system as a whole could translate into enormous potential cost savings and a large impact on customer satisfaction.

Similar to industry practices, the logistics literature has usually treated strategic facility location decisions and tactical inventory management decisions disparately due to the nature of their time horizons; strategic models are considered long-term, and tactical decisions are considered short-term. Hence, supply chain optimization models have typically overlooked the interaction between the facility location, distribution, and inventory management decisions.

In this article, we introduce the capacitated warehouse location model with risk pooling (CLMRP) that incorporates inventory decisions into a location/allocation model. We consider a logistics system in which a single plant ships one type of product to a set of retailers, each with uncertain demand. Distribution centers (DCs)[1] serve as the direct intermediary between the plant and the retailers for the shipment of the product. Safety stock is retained at the DCs to provide appropriate service levels. We assume single sourcing for the retailers, that is, a retailer has to be served by a single DC.

The objective of the CLMRP is to minimize the sum of the fixed facility location, transportation, and inventory

[1] The CLMRP is appropriate for locating distribution centers that maintain working inventory and safety stock inventory such as warehouses rather than those centers that act as transshipment points. In the remainder of the paper, however, the words warehouse and DC are used interchangeably.

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carrying costs. The model simultaneously determines DC locations, shipment sizes and frequencies from the plant to the DCs, the working inventory and safety stock levels at the DCs, and the assignment of retailers to the DCs. The model also ensures that the capacity limitations of the DCs are not exceeded.

The capacity constraint for each DC is defined based on the inventory management policy at that DC. More specifically, the capacity constraint ensures that the maximum possible inventory accumulation at a given DC does not exceed that DC's capacity. The more frequently a DC orders the product from the plant, the lower the inventory levels at that DC and hence the less tight the capacity constraint becomes. This new approach to modeling capacities affords a much broader interpretation of the capacity issue than does the traditional approach such as in the classical capacitated fixed charge location problem (CFLP) and its variants [1, 15, 16, 18, 22].

In the CFLP, the capacity constraint limits the average quantity of products that can flow through a DC. This traditional modeling of capacity can be interpreted as a *yes* or *no* question, meaning can a retailer be assigned to a DC or not. To this end, if a retailer cannot be assigned to any of the open DCs, a new DC may have to be added to the existing network to increase available capacity. The CLMRP improves upon this model by allowing a retailer to be assigned to an existing DC and permitting that DC to order more frequently from the plant. The CLMRP is therefore, capable of evaluating the tradeoff between establishing more DCs to increase system capacity (incurring fixed location costs) versus ordering more frequently from the plant (increasing ordering costs). This new approach to modeling capacities at the warehouses is parallel to the modern view of capacity planning at manufacturing facilities as discussed in [17].

We formulate the CLMRP as a nonlinear integer program. A Lagrangian relaxation solution algorithm is proposed because Lagrangian relaxation algorithms have been successfully developed for traditional facility location problems. However, the objective functions of the traditional models are linear whereas the objective function of the CLMRP is nonlinear and it is neither concave nor convex. The Lagrangian subproblem is also a nonlinear integer program for which we develop an efficient algorithm. The Lagrangian relaxation solution algorithm presented in this article provides near-optimal solutions with reasonable computational requirements for large problem instances.

The article is organized as follows. In Section 2, we review some of the literature related to location theory and inventory models. In Section 3, we present the CLMRP and outline the solution algorithm for the problem, which exploits the structure of the model. Section 4 presents the computational results of our solution algorithms. Section 5 outlines our future research plans.

2. LITERATURE REVIEW

Most research in logistics literature has studied inventory theory and location theory separately. Inventory theory focuses on the evaluation of inventory replenishment strategies at the DCs and the retailers. These studies assume that the strategic location decisions have been made; the number and locations of DCs are assumed to be known. The objective of most inventory models is the minimization of inventory costs while providing appropriate service levels. For a detailed study of inventory models, see texts by Graves et al. [14] and Nahmias [21]. Alternatively, location theory literature focuses on finding the optimal number and locations of DCs and the appropriate DC-retailer assignments. The objective of most location models is the minimization of fixed facility location and transportation costs, ignoring inventory related costs. For a summary of location models, see the texts by Daskin [7] and Drezner [9].

Baumol and Wolfe's [5] work introduces the idea of incorporating inventory costs into location models. The authors argued that inventory costs should add a square root term to the objective function of the uncapacitated fixed charge location problem (UFLP), which is an NP-Hard Problem (see Krarup and Pruzan [19] for a proof of NP hardness of uncapacitated facility location problem). However, they were only able to solve for local optima using a simple heuristic.

A number of integrated location-inventory models have appeared recently. Barahona and Jensen [4] solved a location model with a fixed inventory cost through Dantzig-Wolfe decomposition. Erlebacher and Meller [11] formulated a location-inventory model in which the demand points are continously represented. Teo et al. [31] developed a $\sqrt{2}$-approximation algorithm for a location model that considers inventory costs but ignores transportation costs. Teo and Shu [32] introduced a joint location-inventory model that considers a multiechelon inventory cost function and solved the resulting problem using column generation. Nozick and Turnquist [23] incorporated a safety stock cost term into the traditional location models and solved the resulting model by linearizing the inventory related terms. Nozick and Turnquist [24] also introduced a network design problem for multiple products and determined whether the safety stock for a specific product should be retained at the plant or at the DCs. Eskigun et al. [12] introduced a location model that considers the pipeline inventory costs based on the expected lead time from the plants to the distribution centers. The lead time experienced by a distribution center is formulated as a function of the volume of demand assigned to that distribution center. This model is more appropriate for locating cross-docking centers or for locating distribution centers for which the bottleneck is the server availability at loading/unloading docks rather than the space availability at the distribution centers itself.

Shen et al. [27] and Daskin et al. [8] developed the location model with risk pooling (LMRP) that explicitly incorporated inventory decisions into the UFLP. The authors aimed to capture the "risk pooling effects" in this model. Eppen [10] studied the effects of grouping retailers on inventory costs, namely risk pooling effects. He showed that when each retailer faces independent demands, the total expected safety stock costs are significantly less in the centralized mode than in the decentralized mode. The LMRP made use of this result by keeping safety stock for retailers at their designated warehouses. The inventory related costs add a concave function (an economies of scale cost term) to the objective function of the LMRP. Balcik [3] and Shen [29] studied the multiproduct extension of the LMRP.

All of the location-inventory models mentioned earlier incorporate an economies of scale term into the objective function that captures the risk-pooling effects. In this article, we propose one of the first network design problems, which considers a cost term at each DC that exhibits initial economies of scale and then, after a given increase in demand, exhibits an exponential increase. This enables us to evaluate the tradeoff between risk-pooling benefits and ordering costs incurred at a given DC as more retailers are assigned to that DC. The model also captures the interdependence between inventory levels and capacity limitations (space availability) at the DCs. We describe the model and its properties in the next section.

3. CAPACITATED LOCATION MODEL WITH RISK POOLING

3.1. Model Formulation

In this section, we introduce the CLMRP that models the storage and movement of a single product from a single plant to a set I of retailers through a set J of candidate DCs. We assume direct shipments from the DCs to the retailers and that a retailer has to be served by a single DC. We also assume that the daily demands at the retailers are uncorrelated over time and across retailers and follow Poisson process (Daskin et al. [8], Shen et al. [27], Ozsen et al. [26]). This implies that the variance of the daily demand, σ_i^2, is equal to the mean of daily demand, μ_i for each retailer i.

To meet the demands of the retailers, the DCs hold two types of inventory: the working inventory, which depends on the inventory ordering policy adopted at the DCs, and the safety stock, which is maintained to protect the system against possible stock outs during the replenishment lead time and depends on the service level. When calculating the safety stock cost at the warehouses, we use the Normal approximation to Poisson distribution. Note that approximating a Poisson demand process by Normally distributed demands can be shown to be good for sufficiently large demand values (Montgomery [20]).

We first detail the inventory policy the DCs follow to start modeling the problem. Each DC orders from the plant using an approximation to the (Q,r) model with Type I service (Nahmias [21], pp. 256). We approximate the (Q,r) model using two steps. In the first step, the order quantity is determined using the EOQ model. Then, based on this order quantity, the reorder point is determined. It is well known that this two-step approach provides a good approximation to the optimal order quantity and reorder point values (Axsater [2]). Following this approach, we first solve a capacitated EOQ problem to determine the optimal order quantity.

Let Q_j represent the (unknown) reorder quantity at DC j and r_j represent the reorder point at DC j. Let C_j be the available capacity at DC j, and L_j be the lead time in *days* for deliveries from the plant to the distribution center j. We interpret the capacity constraints to mean that at any point in time the maximum accumulation at a DC should not exceed that DC's capacity. This interpretation has practical value because the amount of space that the warehouse needs is proportional to peak inventory, not annual flow or average inventory as also pointed out by Simchi-Levi et al. [28]. Note that if a DC observes no demand during the replenishment lead time, the inventory levels at that DC would reach its maximum. Hence, we can conclude that the maximum possible accumulation at DC j is the sum of the order quantity, Q_j, and the reorder point, r_j. As a result, under the worst-case scenario (when no demand is observed during the lead time), the capacity constraint can be written as follows:

$$Q_j + r_j \leq C_j$$

where r_j = safety stock + E [demand during lead time]. The major complication in setting up the capacitated EOQ model is that the order quantity, the working inventory, and the safety stock levels at a DC are a function of the total volume of demand assigned to the DC, which is a function of the endogenously determined DC-retailer assignments. To work around this interdependency, assume for the moment that the DC-retailer assignments are known. Let S_j be the set of retailers assigned to DC j, then the lead time demand at DC j has a mean and variance of $L_j \sum_{i \in S_j} \mu_i$. Then, the safety stock required to ensure that stock outs occur with a probability of α or less is $z_\alpha \sqrt{L_j \sum_{i \in S_j} \mu_i}$ where z_α is a standard Normal deviate such that $P(z \leq z_\alpha) = \alpha$.

Let χ represent the number of days in a year and D_j represents the expected annual demand of retailers in S_j, that is: $D_j = \chi \sum_{i \in S_j} \mu_i$. Given the additional notation, the safety stock amount for DC j can be rewritten as $z_\alpha \sqrt{L_j D_j / \chi}$, and the expected demand during the lead time becomes $L_j \frac{D_j}{\chi}$.

Hence, the capacity constraint for DC j under the worst-case scenario is as follows:

$$Q_j + z_\alpha\sqrt{L_j D_j/\chi} + L_j\frac{D_j}{\chi} \le C_j$$

To formulate the annual cost of ordering from the plant at the distribution center, we introduce some more notation. Let F_j be the fixed cost of placing an order from distribution center j to the plant, and let h be the *annual* holding cost per item. The fixed cost of shipping an order to DC j is g_j, and the per-unit shipment cost from the plant to DC j is a_j. To minimize the working inventory costs, we need to solve the following capacitated EOQ problem for DC j, where the reorder quantity, Q_j, is the decision variable:

$$W_j^*(D_j) = \begin{cases} \text{Minimize } G_j(Q_j) = F_j\dfrac{D_j}{Q_j} \\ \qquad + \beta\left(g_j\dfrac{D_j}{Q_j} + a_j D_j\right) + \theta\dfrac{hQ_j}{2} \\ \text{subject to } Q_j + z_\alpha\sqrt{L_j D_j/\chi} + L_j\dfrac{D_j}{\chi} \le C_j \\ \qquad Q_j \ge 0 \end{cases}$$

The objective function of this EOQ model, $H_j(Q_j)$, is the expected annual working inventory cost. The first term is the total fixed cost of ordering Q_j units. The second term represents the transportation cost from the plant to DC j. The last term is the cost of holding an average of $Q_j/2$ units of inventory. β and θ are, respectively, the weight factors associated with the transportation and inventory costs. These weights facilitate the experimental design in which we test the relative importance of transportation and inventory costs with respect to fixed facility location costs.

Let $W_j^*(D_j)$ be the optimal total working inventory cost for DC j, that is the optimal objective value to the nonlinear program above. Note that this cost term is nonlinear and is neither a concave nor a convex function of the total assigned demand and grows asymptotically. The properties of this cost term are briefly summarized in Appendix B and for more details regarding the term, the interested reader is referred to Ozsen [25] and Ozsen et al. [26].

Recall that, the derivation of the optimal working inventory cost for DC j assumes that we know the customer set, S_j, assigned to DC j. So, we need to solve for set S_j endogenously as well. To simultaneously determine the locations of the DCs, the DC-retailer assignments and the working and safety stock inventory costs, we define the following decision variables:

$$X_j = \begin{cases} 1, & \text{if we locate at candidate site } j \\ 0, & \text{if not} \end{cases}$$

$$Y_{ij} = \begin{cases} 1, & \text{if demands at customer } i \text{ are assigned to a DC at candidate site } j, \\ 0, & \text{if not} \end{cases}$$

Given the assignment variables, D_j can be rewritten as $\sum_{i\in I}\chi\mu_i Y_{ij}$. So far, we have only discussed the costs associated with the orders from the plant to the DCs. A fixed location cost, f_j, is incurred to establish a DC j. Transportation costs are incurred for the direct shipment of the product from each DC to its customer set. Let d_{ij} be the cost per unit to ship from candidate DC site j to retailer i. Now, we can formulate the CLMRP as follows:

$$\begin{aligned} \text{Minimize} \quad & \sum_{j\in J}\left[f_j X_j + \beta\chi\sum_{i\in I} d_{ij}\mu_i Y_{ij} + \theta h z_\alpha\sqrt{L_j}\sqrt{\sum_{i\in I}\mu_i Y_{ij}}\right] \\ & + \sum_{j\in J}\left[(F_j + \beta g_j)\frac{\chi\sum_{i\in I}\mu_i Y_{ij}}{Q_j} + \beta\chi\sum_{i\in I} a_j\mu_i Y_{ij} + \theta\frac{hQ_j}{2}\right] \qquad (1) \\ \text{Subject to} \quad & \sum_{j\in J} Y_{ij} = 1, \quad \forall i \in I \qquad (2) \\ & Y_{ij} - X_j \le 0, \quad \forall i\in I, j\in J \qquad (3) \\ & Q_j + \left(z_\alpha\sqrt{L_j}\sqrt{\sum_{i\in I}\mu_i Y_{ij}} + L_j\sum_{i\in I}\mu_i Y_{ij}\right) \le C_j, \quad \forall j\in J \qquad (4) \\ & Q_j \ge 0, \quad \forall j \in J \qquad (5) \\ & Y_{ij}\in\{0,1\}, \quad \forall i\in I, j\in J \qquad (6) \\ & X_j \in\{0,1\}, \quad \forall j\in J \qquad (7) \end{aligned}$$

The objective function (1) sums the fixed cost of locating distribution centers, the DC-retailer transportation cost, the safety stock cost, and the working inventory cost. Constraints (2) require that each retailer is assigned to exactly one distribution center. Constraints (3) state that retailers can only be assigned to open DCs. (4) and (5) are constraints of the capacitated EOQ problem discussed earlier. (6) and (7) are standard binary constraints.

This model has nonlinear terms in both the objective function and the constraints unlike the UFLP or the LMRP. In addition, to obtain the optimal solution, we need to solve for the order quantity variables explicitly unlike in the LMRP. To simplify matters, instead of representing the working inventory cost and the reorder quantity explicitly, we include $W_j^*(D_j)$ in the formulation above. To simplify the notation further, let $\bar{W}_j^*(\chi\sum_{i\in I}\mu_i Y_{ij})$ denote the optimal working inventory and safety stock cost for DC j, as a function of the expected assigned annual demand. That is

$$\bar{W}_j^*\Big(\chi\sum_{i\in I}\mu_i Y_{ij}\Big) = W_j^*\Big(\chi\sum_{i\in I}\mu_i Y_{ij}\Big) + \theta h z_\alpha\sqrt{L_j}\sqrt{\sum_{i\in I}\mu_i Y_{ij}}$$

In other words, we choose to work with the following equivalent CLMRP formulation for the remainder of the paper:

$$(\textbf{CLMRP})\ \text{Minimize} \quad \sum_{j\in J}\Big[f_j X_j + \beta\chi\sum_{i\in I} d_{ij}\mu_i Y_{ij} + \bar{W}_j^*\Big(\chi\sum_{i\in I}\mu_i Y_{ij}\Big)\Big] \tag{8}$$

$$\text{subject to} \quad \sum_{j\in J} Y_{ij} = 1, \quad \forall i \in I \tag{9}$$

$$Y_{ij} - X_j \le 0, \quad \forall i \in I, j \in J \tag{10}$$

$$Y_{ij} \in \{0,1\}, \quad \forall i \in I, j \in J \tag{11}$$

$$X_j \in \{0,1\}, \quad \forall j \in J \tag{12}$$

We prefer this formulation for two reasons. First, we can solve for the optimal order quantity endogenously. Second, this formulation allows the optimal working inventory cost function, $\bar{W}_j^*(D_j)$, embedded in the objective function to be due to any inventory policy not only the one described in this section. In this sense, this formulation is more general. Also, note that if it were not for the third term in the objective function, the CLMRP formulation would be identical to the classical UFLP formulation. This term has the same properties as those of $W_j^*(D_j)$, which are discussed briefly in Appendix B.

Before concluding this section, we note that the capacity constraint and the optimal working inventory cost function are derived under the worst case scenario. In reality, the probability of observing no demand during a lead time should be very low. In this sense, the capacity constraint is very conservative. However, we can solve the chance constraint variant of the problem (see Appendix C for more details). If we construct the capacity constraint considering that the probability of inventory accumulation exceeding the capacity is low, then we lose the linear term in the capacity constraint (4). Instead, a probability-related constant is added as a coefficient of D_j in the square root term. Although this seems to change the structure of the constraint, the essential properties of the optimal working inventory cost function remain the same. Hence, the solution approach we develop in the next section applies to the case in which DC capacity limitations are modeled as chance constraints as well.

4. SOLUTION APPROACH

4.1. Obtaining a Lower Bound

As we discussed earlier, Lagrangian relaxation-based heuristics have been developed successfully for traditional location models. Fisher [13] provides an excellent discussion of Lagrangian relaxation. For its application to the UFLP and the CFLP see Daskin [7; pp. 247–302]. Because the CLMRP is a variant of the UFLP, we choose to develop a Lagrangian Relaxation Algorithm to solve the CLMRP. By relaxing the assignment constraints, we obtain the following Lagrangian Dual problem:

$$\begin{aligned}\text{Max}_\pi\ \text{Min}_{X,Y} &\sum_{j\in J}\Big[f_j X_j + \beta\chi\sum_{i\in I} d_{ij}\mu_i Y_{ij} + \bar{W}_j^*\Big(\chi\sum_{i\in I}\mu_i Y_{ij}\Big)\Big] + \sum_{i\in I}\pi_i\Big(1-\sum_{j\in J} Y_{ij}\Big)\\ &= \sum_{j\in J}\Big[f_j X_j + \sum_{i\in I}(\beta\chi d_{ij}\mu_i - \pi_i)Y_{ij} + \bar{W}_j^*\Big(\chi\sum_{i\in I}\mu_i Y_{ij}\Big)\Big] + \sum_{i\in I}\pi_i\end{aligned} \tag{13}$$

$$Y_{ij} - X_j \le 0, \quad \forall i \in I, j \in J \tag{14}$$

$$Y_{ij} \in \{0,1\}, \quad \forall i \in I, j \in J \tag{15}$$

$$X_j \in \{0,1\}, \quad \forall i \in I, j \in J \tag{16}$$

For fixed values of the Lagrangian multipliers, π, we want to minimize (13) over the location variables, X_j, and the assignment variables, Y_{ij}. For a given π vector, the problem decouples to the following subproblem for each distribution center j:

$$(\textbf{C-SP}_\mathbf{j}) = \begin{cases}\text{Minimize} & V_j = f_j + \sum_{i\in I}(\beta\chi d_{ij}\mu_i - \pi_i)Y_{ij} + \bar{W}_j^*\Big(\chi\sum_{i\in I}\mu_i Y_{ij}\Big)\\ \text{Subject to} & Y_{ij} \in \{0,1\}, \forall i \in I\end{cases}$$

V_j denotes the benefit of facility j and represents the contribution of opening facility j to the objective function (13). To obtain a lower bound for the CLMRP, we set $X_j = 1$ if $V_j \le 0$, otherwise we set $X_j = 0$. If in the optimal solution to (**C-SP$_\mathbf{j}$**), $X_j = 1$, and $Y_{ij} = 1$ then we set $Y_{ij} = 1$ in (13)–(16), otherwise we set $Y_{ij} = 0$ in (13)–(16).

For Lagrangian relaxation to be effective, the subproblem must be solved efficiently. If the objective function of the subproblem (**C-SP$_\mathbf{j}$**) was linear, that is, if there was no working inventory and safety stock cost term as in the UFLP, the subproblem could be solved simply by algebraic computation [7; pp. 250–274]. If the optimal working inventory plus the safety stock cost term in the objective function, $\bar{W}_j^*(D_j)$, was concave then there is an $O(n\log n)$ time algorithm due to Shen et al. [27]. However, $\bar{W}_j^*(D_j)$, is neither concave nor convex. Hence, solving the subproblem is not straightforward. We begin the analysis of the problem ($\mathbf{C\text{-}SP_j}$) with an observation based on its integrality relaxation. The relaxed problem ($\mathbf{C\text{-}SP_j^{RP}}$) is

$$(\mathbf{C\text{-}SP_j^{RP}}) = \begin{cases} \text{Minimize} & V_j^{RP} = f_j + \sum_{i\in I} A_i Y_{ij} \\ & \quad + \bar{W}_j^*\left(\chi \sum_{i\in I} \mu_i Y_{ij}\right) \\ \text{Subject to} & 0 \le Y_{ij} \le 1, \forall i \in I \end{cases}$$

where $A_i = \beta\chi\mu_i d_{ij} - \pi_i$

THEOREM 1: Assuming that the set of retailers I has been sorted such that:

$$\frac{A_1}{\mu_1} \le \frac{A_2}{\mu_2} \le \cdots \le \frac{A_m}{\mu_m}$$

where $m = |I|$.

Then, there is an optimal solution, $\mathbf{Y_{ij}^*}$ to the ($\mathbf{C\text{-}SP_j^{RP}}$) with the following two properties:

1. At most one of the assignment variables Y_{ij}^*, takes on a fractional value;
2. If $Y_{kj}^* > 0$, for some $k \in 1, \dots, m$, then $Y_{ij}^* = 1$, $\forall i \in \{1, \dots, k-1\}$.

PROOF: See Appendix A. □

To solve the relaxed subproblem, first we need to be able to determine if there is an assignment variable with fractional optimal value. And if there is one, we need to compute its optimal value. To this end, we prove the following theorem:

THEOREM 2: Assume that $\mathbf{Y_{ij}^*}$ is an optimal solution to ($\mathbf{C\text{-}SP_j^{RP}}$). If assignment variable Y_{kj}^* takes on a fractional value, then for that retailer k the following condition holds:

$$A_k = -\frac{\partial \bar{W}_j^*\left(\chi \sum_{i\in I} \mu_i Y_{ij}^*\right)}{\partial Y_{kj}^*}$$

PROOF: See Appendix A. □

Theorem 2 essentially states that the marginal inventory cost of including retailer i in the customer set of DC j should equal to the shipment cost (modified by the lagrange multiplier, π_i) from DC j to retailer i. Now to simplify the notation, we can rewrite this condition as follows:

By chain rule:
$$\frac{\partial \bar{W}_j^*\left(\chi \sum_{i\in I} \mu_i Y_{ij}^*\right)}{\partial Y_{kj}^*} = \frac{\partial \bar{W}_j^*\left(\chi \sum_{i\in I} \mu_i Y_{ij}^*\right)}{\partial\left(\chi \sum_{i\in I} \mu_i Y_{ij}^*\right)} \times \frac{\partial\left(\chi \sum_{i\in I} \mu_i Y_{ij}^*\right)}{\partial Y_{kj}^*}$$

By letting
$$\bar{E}_j^*\left(\chi \sum_{i\in I} \mu_i Y_{ij}^*\right) = \frac{\partial \bar{W}_j^*\left(\chi \sum_{i\in I} \mu_i Y_{ij}^*\right)}{\partial\left(\chi \sum_{i\in I} \mu_i Y_{ij}^*\right)}$$

Then we have
$$\frac{\partial \bar{W}_j^*\left(\chi \sum_{i\in I} \mu_i Y_{ij}^*\right)}{\partial Y_{kj}^*} = \bar{E}_j^*\left(\chi \sum_{i\in I} \mu_i Y_{ij}^*\right)\chi\mu_k$$

Hence, if there is an assignment variable, Y_{ij}, that takes on a fractional value in the optimal solution, the following condition holds for that retailer i:

$$A_k + \chi\mu_k \bar{E}_j^*\left(\chi \sum_{i\in I} \mu_i Y_{ij}^*\right) = 0 \tag{17}$$

Note that Theorem 1 does not make any assumptions about the inventory and safety stock cost function that appear in the objective function of ($\mathbf{C\text{-}SP_j^{RP}}$). Theorem 2 assumes that this cost function is differentiable. Hence, these properties hold for any differentiable $\bar{W}_j^*(D_j)$, not just the one described and derived in this article. In Ozsen [25] and Ozsen et al. [26] we derive this function and show that it is differentiable. Based on these two theorems, we propose the following algorithm to solve the problem ($\mathbf{C\text{-}SP_j^{RP}}$):

ALGORITHM_C-SP$_j^{RP}$:

STEP 1: Sort the elements of I such that

$$\frac{A_1}{\mu_1} \le \frac{A_2}{\mu_2} \le \cdots \le \frac{A_m}{\mu_m}$$

where $m = |I|$.

Use this ordering of the elements of $|I|$ in the following steps.

STEP 2: Let $D_{j,k} = \chi \sum_{i=1}^{k} \mu_i$. For $k = 1, \dots, m$:

Define set $\Delta_{j,k} = \{\hat{D}_{j,k} \in \Re \mid A_k + \chi\mu_k \bar{E}_j^*(\hat{D}_{j,k}) = 0 \text{ and } D_{j,k-1} < \hat{D}_{j,k} < D_{j,k}\}$

This step determines whether Y_{kj} can take on a fractional value.

CASE 1: If set $\Delta_{j,k} = \emptyset$ compute the partial sum as follows:

$$S_{j,k} = \sum_{i=1}^{k} A_i + \bar{W}_j^*(D_{j,k})$$

If Y_{kj} cannot take on a fractional value, then we serve retailer k's entire demand.

CASE 2: If set $\Delta_{j,k} \neq \emptyset$ then:

$$S_{j,k} = \text{Min}\left\{\hat{S}_{j,k}, \sum_{i=1}^{k} A_i + \bar{W}_j^*(D_{j,k})\right\}$$

where the first term $\hat{S}_{j,k} = \text{Min}_{\hat{D}_{j,k} \in \Delta_{j,k}} \{\sum_{i=1}^{k-1} A_i + A_k(\hat{D}_{j,k} - D_{j,k-1})/\chi\mu_k + \bar{W}_j^*(\hat{D}_{j,k})\}$.

Record the $D_{j,k}^*$ that gives the $S_{j,k}$ value.

The first term in $S_{j,k}$ is the cost due to serving a fraction of retailer k's demand. The second term is due to the integral solution, which results from serving the entire demand of retailer k. We take the minimum of these two terms to minimize the cost. Note that Step 2 is repeated for all retailers in set I.

STEP 3: Let k^* be the value of k that gives the minimum $S_{j,k}$ value of Step 2.

(a) Due to Theorem 1:

$$Y_{ij} = \begin{cases} 0, & \text{if } i > k^* \\ 1, & \text{if } i < k^* \end{cases}$$

(b) Due to Theorem 2: For $i = k^*$, $Y_{ij} = (D_{j,k^*}^* - D_{j,k^*-1})/\chi\mu_{k^*}$ (Note that $Y_{ij} = 1$ if $S_{j,k^*} \neq \hat{S}_{j,k^*}$)

Algorithm_C-SP$_j^{\text{RP}}$ works not only for $\bar{W}_j^*(D_j)$ due to the capacitated EOQ model that we discussed in the previous section, but also for any differentiable expected total working inventory cost function of D_j. This algorithm would take $O(|I|log(|I|)$ if Step 2 did not involve finding the "zeros" of a function. Therefore, the efficiency of the algorithm depends on the behavior of the $\bar{W}_j^*(D_j)$ function. Note that we use a bisection search procedure to find the zeros this function.

The $\bar{W}_j^*(D_j)$ is also shown to be monotone increasing in Ozsen [25] and Ozsen et al. [26]. And since the subproblem (**C-SP$_\mathbf{j}^{\mathbf{RP}}$**) is a minimization problem, the retailers that belong to the set $I^+ = \{i \in I : A_i > 0\}$ cannot be served by DC j in the optimal solution. Therefore, we can limit the search space to $I^- = \{i \in I : A_i \leq 0\}$ rather than the set of all retailers I in the first Step of the Algorithm_C-SP$_j^{\text{RP}}$. Note that this additional observation does not improve the complexity of the algorithm but could potentially reduce the computational effort.

An important special case is when $\bar{W}_j^*(D_j)$ in subproblem (**C-SP$_\mathbf{j}^{\mathbf{RP}}$**) is concave. Since then, the subproblem becomes a concave minimization problem, the assignment variables will have the integrality property [30]. In that case, the algorithm takes $O(|I|log(|I|)$ as Case 2 of Step 2 can be skipped. This special case is given in Shen [27] for the LMRP.

We solve the subproblem (**C-SP$_\mathbf{j}^{\mathbf{RP}}$**) using the improved version of Algorithm_C-SP$_j^{\text{RP}}$ in which we limit the search space to $I^- = \{i \in I : A_i \leq 0\}$ and compute the benefit of a facility, $V_j^{\text{RP}} = f_j + S_{j,k^*}$. The solution of (**C-SP$_\mathbf{j}^{\mathbf{RP}}$**) provides a lower bound solution for (**C-SP$_\mathbf{j}$**) and hence can be used to find a lower bound solution for the original problem, the CLMRP. We set $X_j = 1$ if $V_j^{\text{RP}} \leq 0$, otherwise we set $X_j = 0$. Then if $X_j = 1$, we set Y_{ij} to the appropriate values in (13)–(16) as outlined in Step 3 of algorithm; otherwise we set $Y_{ij} = 0$ in (13)–(16).

Solving the Lagrangian Dual Problem as described earlier for given values of π_i provides a lower bound on the optimal value of the CLMRP. To find the optimal Lagrange multipliers, we use a standard subgradient optimization procedure [13]. At each step of the Lagrangian heuristic, we use the dual-feasible solution to construct a primal feasible solution to the CLMRP, which provides an upper bound. We update the Lagrangian multipliers, π_i, using subgradient optimization and proceed to the next iteration. We repeat this procedure until the bounds converge or until some stopping criterion is reached. The details of how to construct a primal feasible solution are described in the next section.

The resulting lower bound solution due to (**C-SP$_\mathbf{j}^{\mathbf{RP}}$**), however, is not guaranteed to provide a tight bound for the CLMRP. Hence, we apply branch and bound on the assignment variables. We have shown in Theorem 1 that at most one assignment variable can take on a fractional value in the optimal solution of (**C-SP$_\mathbf{j}^{\mathbf{RP}}$**). So, at each node of the branch bound tree, we branch on the retailer with the fractional assignment value, if there is one. At every stage of this procedure, a retailer is first forced to be in the customer set served by the DC and then forced out of the customer set. Branching is done in a depth-first manner.

The fact that $\bar{W}_j^*(D_j)$ is monotone increasing allows us also to improve the branching procedure as described in the following observations and theorems.

OBSERVATION 1: Fixing the retailers does not change the structure of the subproblem (**C-SP$_\mathbf{j}^{\mathbf{RP}}$**) and the algorithm steps.

PROOF: When a retailer is forced out of the solution at some node of the branch and bound tree, the retailer is removed from the set I^- at that node and the children of the node. When a retailer is forced to be in the customer set served by a DC at a node of the branch and bound tree, the cost associated with serving that retailer is part of the fixed cost of opening that DC. The retailer that is forced in is also removed from the set I^- at that node and the children of that node. □

THEOREM 3: Assume that the subset of retailers $I^- = \{i \in I : A_i \leq 0\}$ has been sorted such that:

$$\frac{A_1}{\mu_1} \leq \frac{A_2}{\mu_2} \leq \cdots \leq \frac{A_m}{\mu_m}$$

If retailer k is forced out at node t of the branch and bound tree, i.e. $Y_{kj} = 0$ and for some $l > k$ if $A_l > A_k$ and $\mu_l > \mu_k$ then retailer l can be also forced out at node t of the branch and bound tree (meaning, there is an optimal integer solution with $Y_{lj} = 0$.)

PROOF: This follows from the fact that the $\bar{W}_j^*(\chi \sum_{i \in I} \mu_i Y_{ij})$ is monotone increasing. See Appendix A for the details of the proof. □

COROLLARY 1: If forcing Y_{kj} out at some node t of the branch and bound tree causes $Y_{lj} = 0$ at node t but Y_{lj} was forced into the solution in a parent node of t, then we can prune node t.

Theorem 3 and Corollary 1 allow us to decrease the number of nodes on the branch and bound tree when solving the subproblem (**C-SP$_\mathbf{j}$**).

4.2. Obtaining an Upper Bound

At each iteration of the Lagrangian procedure, we make use of the current lower bound solution $(X^{\mathrm{LB}}, Y^{\mathrm{LB}})$ to obtain a feasible solution for the CLMRP. First, we open a DC j only if $X_j^{\mathrm{LB}} = 1$ in the current Lagrangian solution. Then we sort the retailers in decreasing order of mean demand, μ_i, and loop through the retailers based on this order. We first process the retailers with $\sum_{j \in J} Y_{ij}^{\mathrm{LB}} > 0$ and assign retailer i to DC j with $Y_{ij}^{\mathrm{LB}} > 0$ that increases the total cost the least, based on the assignments made so far. Next, we process the retailers with $\sum_{j \in J} Y_{ij}^{\mathrm{LB}} = 0$ and assign retailer i to any open DC that increases the total cost the least. At the end of this process, if there are DCs with no retailers assigned to them, then we close those DCs. If all the retailers are assigned to some DC, then the resulting solution is feasible for the CLMRP and provides an upper bound on the objective function (8).

It is possible to have some retailers not assigned to any DC if there are not enough open DCs with sufficient capacity in the Lagrangian solution. To avoid this possibility at the first Lagrangian iteration, we apply a modified heuristic. At the first Lagrangian iteration, we force open all the DCs and process the retailers in decreasing order of mean demand and assign each retailer to the DC that increases the total cost the least.

At the end of the Lagrangian procedure, if the lower bound is strictly less than the upper bound obtained, we apply a DC-exchange heuristic. The heuristic swaps a DC currently open in the solution with another DC that is not currently in the solution, if doing so improves the solution. This procedure is a variant of Teitz and Bart's [33] procedure for the p-median problem. We observed that this heuristic improves the objective function value for about 60% of the problem instances, and the improvement is more significant for those problems that have tighter capacity constraints.

5. COMPUTATIONAL EXPERIMENTS AND RESULTS

In this section, we first explain the design of our experiments and then summarize the computational results. We tested our algorithm for the CLMRP on a total of 88 instances generated using four different data sets: a 15-node, a 49-node, an 88-node data, and a 150-node data set. The 15-node data set consists of the 15 nodes with the highest demand from the 49-node data set, which is described in Daskin [7; pp. 480–482]. The 49-node data set represents the capitals of the lower 48 United States plus Washington, DC. The 88-node data set contains the 49-node data set plus the 50 largest cities in the 1990 U.S. census (with duplicates eliminated), and the 150-node data set contains the 150 largest cities in the 1990 U.S. census as described in Daskin [7; pp. 476–479]. Each of the nodes represents a retail location. In all the experiments, each retail location was also a candidate DC location.

Fixed facility location costs were obtained by dividing the facility location costs in Daskin [7; pp. 476–482] by 100. Note that for the 150-node data set, the fixed facility locations costs are the same for all the candidate DC sites. We set the unit cost of shipping from candidate DC j to retailer i, d_{ij}, to the great circle distance between these locations. The fixed ordering F_j and shipping g_j costs were set to 10, and the variable shipping cost a_j was set to 5 for all DCs. The holding cost was set to be 1, and z_α was set to be 1.96 corresponding to a 97.5 % service level. The lead time, L, and the days per year χ were set to 1. Although $\chi = 1$, the difference between the daily parameters and yearly parameters are realized through the weights β and θ. The weight β appears wherever χ appears in the objective function (8). Similarly, the weight θ appears wherever $\frac{L}{\chi}$ appears in the objective function (see also the details of the $\bar{W}_j^*(D_j)$ function in Appendix B). Hence, varying these two weights allow us to experiment with different

Table 1. Parameters for the Lagrangian relaxation procedure.

Parameter	Value
Maximum number of iterations at each node	1200
Number of number of iterations before halving α	12
Initial value of α	2
Minimum value of α	0.00000001
Minimum LB-UB gap	0.001%
Initial value for π_i	$10\bar{\mu} + 10f_i$

values of L and χ in addition to all the other transportation and inventory related parameters.

The parameters for Lagrangian relaxation used for all the experiments are given in Table 1. The notation $\bar{\mu}$ in Table 1 stands for the average mean demand across all retailers. We branch on Y_{ij} for those Lagrangian iterations greater than 10. We terminated the Lagrangian procedure based on the optimality gap, or the maximum number of iterations allowed or the minimum value of α (scalar used in calculating the step size), whichever occured first. In none of the experiments, the procedure reaches the maximum number of iterations. The optimality gap is defined as (upper bound − lower bound) × 100/lower bound. Times obtained for our algorithm are on a Dell Inspiron running at 1.7 GHz using Windows XP. The program was written in C++.

For all four data sets, the mean demand was obtained by dividing the population data given in Daskin (1995) by 1000. To vary the difficulty of problem instances, we first set β and θ values to 0.0004 and 0.01 and then to 0.00001 and 0.001, respectively, which in effect decreases the relative ratio of the weights on transportation costs over the weights on inventory costs from 0.04 to 0.01, in addition to making the fixed facility costs relatively more important. We refer to $(\beta, \theta) = (0, 0004, 0.01)$ pair as weight pair 1 and to $(\beta, \theta) = (0, 00001, 0.001)$ pair as weight pair 2 in the remainder of this section.

In addition to varying the weights, we have tested different values of the DC capacities to vary the difficulty of the instances. We first set the capacities of the DCs to large values and solve the CLMRP, which is equivalent to solving the LMRP. Then, we limit the capacity of one additional DC at a time to obtain the next problem instance. For example, to have one DC with limited capacity in the next instance, we set the capacity of the DC with the most assigned demand in the optimal solution to the LMRP to half the value of the assigned demand. We repeat this process in a similar fashion as follows: let DC j be the facility with the most assigned demand, D_j in the optimal solution to test problem k, then in test problem $k + 1$, the capacity of DC j is set to $D_j/2$ to increase the capacity tightness in test problem $k + 1$. This process is illustrated for the case of the 88-node data set for weight pair 2 in Table 2. The first column titled "Capacitated DCs" shows the DCs with limited capacities in the problem instance. The second column, "Open DCs," shows the DCs open in the optimal solution to the problem instance. The third column, "DC w/ MaxD," shows the DC serving the largest volume of demand in the optimal solution. As can be seen from Table 2, in the optimal solution, it is possible to have some DCs with limited capacity open even though there are many other DC candidates with unlimited capacity.

Using the process outlined earlier for defining capacities, we generate 10 instances with varying capacity tightness for each data set (i.e., an instance may consist of up to 9 capacitated DCs). We also test using both weight pairs, resulting in a total of 80 instances. The details of the solutions are presented in Tables 3–6 in which the columns are as follows:

Prob no.: Problem number.
No. rets: Number of retailers in the problem instance.
DCs opened: The additional DCs that are located compared to the previous instance.
DCs closed: The additional DCs that are closed compared to the previous instance.
No. of open DCs: Total number of DCs that are open.
Last capacitated DC: Additional DC that is assigned a limited capacity.
Open DCs with limited capacity: DCs with limited capacity open in the solution.
Total cost: Objective value of the best feasible solution (Upper bound solution).
% Gap: Percentage gap between upper bound and lower bound solution.
Lag iter: Total number of Lagrangian relaxation iterations.
B&B iter: Total number of branch and bound iterations on the assignment variables.
CPU time (s): The number of CPU seconds elapsed before the algorithm terminated.

Table 2. 88-Node data set, $\beta = 0.00001$ and $\theta = 0.001$.

Capacitated DCs	Open DCs	DC w/ MaxD
None	St Louis, Fresno, Harrisburg	Harrisburg
Harrisburg	Detroit, Tucson	Detroit
Harrisburg, Detroit	Philadelphia, Detroit, Tucson	Tucson
Harrisburg, Detroit, Tucson	Philadelphia, Detroit, El Paso	El Paso
Harrisburg, Detroit, Tucson, El Paso	Philadelphia, St. Louis, Fresno	St. Louis

Table 3. Results for 15- and 49-node data sets; $\beta = 0.0004$ and $\theta = 0.01$.

Prob no.	No. rets	DCs opened	DCs closed	No. of open DCs	Last capacitated DC	Open DCs with limited capacity	Total cost	% Gap	Lag iter	B&B iter	CPU time (s)
1	15	1,2,3,4,5,6,7,8,9,10, 11,13,14,15	N/A	14	N/A	N/A	1,782,920	0.0000	4	0	0
2	15	None	1	13	1	None	19,069,501	0.0000	29	45	1
3	15	None	None	13	3	3	20,415,087	0.0000	26	57	1
4	15	None	None	13	15	3,15	21,878,997	0.0000	24	89	1
5	15	None	None	13	6	3,15,6	24,069,591	0.0000	33	233	2
6	15	None	None	13	14	3,15,6,14	24,771,747	0.0000	21	97	1
7	15	None	None	13	8	3,15,6,14,8	26,009,707	0.0000	33	334	3
8	15	None	None	13	7	3,15,6,14,8,7	26,414,628	0.0000	31	355	3
9	15	None	None	13	11	3,15,6,14,8,7,11	27,545,258	0.0000	22	233	2
10	15	None	None	13	4	3,15,6,14,8,7,11,4	29,557,039	0.0000	22	317	3
11	49	1,2,3,4,5,6,7,8,9,10, 11,12,13,14,15,16, 17,18,19,20,21,22, 23,24,25,26,28,29, 30,31,32,33,34,35, 36,37,41,43,44	N/A	39	N/A	N/A	3,764,452	0.0000	41	0	2
12	49	39	1	39	1	None	4,901,703	0.0000	71	165	3
13	49	None	None	39	39	39	8,961,005	0.0000	111	555	7
14	49	None	None	39	41	39,41	9,020,282	0.0000	93	555	7
15	49	None	None	39	29	39,41,29	9,996,539	0.0107	1111	12, 462	118
16	49	None	None	39	35	39,41,29,35	10,722,394	0.0000	91	816	9
17	49	None	None	39	18	39,41,29,35,18	11,186,739	0.0953	1077	17,598	162
18	49	None	None	39	24	39,41,29,35,18,24	12,425,114	0.0037	1148	21,945	200
19	49	None	None	39	43	39,41,29,35,18,24,43	14,139,164	0.0067	1087	24,483	219
20	49	None	None	39	37	39,41,29,35,18,24,43,37	14,317,486	0.0000	174	3,300	30

N/A, not applicable.

Table 4. Results for 88- and 150-node data sets; $\beta = 0.0004$ and $\theta = 0.01$.

Prob no.	No. rets	DCs opened	DCs closed	No. of open DCs	Last capacitated DC	Open DCs with limited capacity	Total cost	% Gap	Lag iter	B&B iter	CPU time (s)
21	88	1,3,4,5,7,9,10,12, 13,15,18,22,23, 24,26,28,30,32, 36,41,51,67	N/A	22	N/A	N/A	2,741,843	0.0010	400	0	10
				22							
22	88	72	1,5	21	1	None	2,765,198	0.0014	1,195	3,443	50
23	88	5	72	21	72	None	2,811,730	0.0000	354	1,974	27
24	88	63	None	22	5	5	2,940,904	0.0008	355	3,144	37
25	88	70	63	22	63	5	3,037,185	0.1024	1,277	15,148	152
26	88	83	12,70	21	70	5	3,041,146	0.0223	1,202	17,576	172
27	88	12,61	83	22	83	5	3,064,111	0.0083	1,308	23,136	221
28	88	75	61	22	61	5	3,080,628	0.2114	1,308	26,489	251
29	88	None	75	21	75	5	3,081,234	0.0390	1,152	26,767	265
30	88	82	12	21	12	5	3,189,939	0.0114	1,255	33,391	307
31	150	1,2,3,4,7,8,24,26, 30,43,51,73,91,94, 101,106	N/A	16	N/A	N/A	3,565,142	0.0000	490	0	20
32	150	65	1	16	1	None	3,582,290	0.0000	596	2,063	40
33	150	54	65	16	65	None	3,596,744	0.0000	493	3,343	49
34	150	122	54	16	54	None	3,621,432	0.0000	395	4,142	55
35	150	81	122	16	122	None	3,623,247	0.0000	355	4,725	64
36	150	121	81	16	81	None	3,736,564	0.0000	445	12,936	137
37	150	5,136	121	17	121	None	3,784,375	0.0002	502	11,017	118
38	150	None	136	16	136	None	3,798,419	0.0456	1,179	26,298	269
39	150	125	None	17	5	5	3,867,545	0.0019	1,203	31,742	310
40	150	108	125	17	125	5	3,936,029	0.0000	687	26,232	248

N/A, not applicable.

Table 5. Results for 15- and 49-node data sets; $\beta = 0.00001$ and $\theta = 0.001$.

Prob no.	No. rets	DCs opened	DCs closed	No. of open DCs	Last capacitated DC	Open DCs with limited capacity	Total cost	% Gap	Lag iter	B&B iter	CPU time (s)
41	15	1,3,4,5,14	N/A	5	N/A	N/A	567,564	0.000	57	0	1
42	15	9	5	5	5	None	595,707	0.000	85	1,154	11
43	15	2,5	9	6	9	5	621,764	0.000	82	2,251	20
44	15	8	14	6	14	5	630,051	0.001	101	4,207	37
45	15	6	8	6	8	5	630,976	0.000	69	3,589	33
46	15	7	6	6	6	5	642,722	0.626	1,074	68,943	618
47	15	6,8	7	7	7	5,6,8	657,981	0.997	1,063	83,033	743
48	15	9,11	2,4	7	2	5,6,8,9	661,070	0.000	309	23,419	197
49	15	4	11	7	11	5,6,8,9	668,430	1.099	1,082	96,989	813
50	15	None	1	6	1	5,6,8,9	987,298	0.738	1,091	104,553	896
51	49	1,3,5,6,22	N/A	5	N/A	N/A	874,678	0.001	324	0	3
52	49	8,9,30	5,6	6	5	None	899,824	0.001	250	16,355	139
53	49	2,5	9	7	9	5	910,817	0.515	1,092	123,303	1,054
54	49	39	1	7	1	5	916,486	0.512	1,073	135,468	1,142
55	49	29	39	7	39	5	966,063	0.486	1,092	148,938	1,268
56	49	6,41	29,30	7	29	5	976,872	0.514	1,147	150,301	1,295
57	49	35	8,41	6	41	5	1,002,017	0.302	1,159	168,188	1,437
58	49	8,30	6	7	6	5	1,005,712	0.600	1,205	163,814	1,387
59	49	18	35	7	35	5	1,026,224	0.457	1,190	167,031	1,419
60	49	24,29	18	8	18	5,29	1,060,598	0.614	1,204	194,414	1,630

N/A, not applicable.

Table 6. Results for 88- and 150-node data sets; $\beta = 0.00001$ and $\theta = 0.001$.

Prob no.	No. rets	DCs opened	DCs closed	No. of open DCs	Last capacitated DC	Open DCs with limited capacity	Total cost	% Gap	Lag iter	B&B iter	CPU time (s)
61	88	34,46,75	N/A	3	N/A	N/A	322,627	0.001	168	0	3
62	88	7,33	34,46,75	2	75	None	327,230	0.001	186	8,361	71
63	88	5	None	3	7	7	328,702	0.025	1,117	129,302	1,099
64	88	22	33	3	33	7	328,808	0.000	303	37,400	329
65	88	34,46	7,22	3	22	None	329,024	0.049	1,042	490,004	3,962
66	88	7,10	34	4	34	7	330,900	0.001	224	79,566	696
67	88	29,40	5,7,10	3	5	None	333,440	0.001	139	61,613	553
68	88	10,23	29,40	3	40	None	337,911	0.001	193	87,412	765
69	88	7,12	23	4	23	7	342,219	0.599	1,051	1,612,263	12,714
70	88	29,75	10,12	4	12	7,75	344,845	0.010	1,113	808,812	6,649
71	150	16,135	N/A	2	N/A	N/A	468,645	0.000	239	0	10
72	150	43	16	2	16	None	469,599	0.006	1,069	72,513	656
73	150	86	43	2	43	None	469,740	0.001	430	95,129	824
74	150	67	86	2	86	None	471,320	0.000	258	105,727	862
75	150	39,134	67,135	2	67	None	473,743	0.001	297	177,718	1,457
76	150	68,135	39,134	2	39	None	474,475	0.000	343	233,836	1,912
77	150	56	68	2	68	None	474,750	0.001	372	307,282	2,485
78	150	13	56	2	56	None	476,508	0.037	1,076	849,127	6,896
79	150	23	13	2	13	None	477,314	0.001	407	462,340	3,783
80	150	95	23	2	23	None	478,615	0.000	266	312,736	2,536

N/A, not applicable.

Note that the first row for each data set in these four tables provides the solution for the base case where there are no limited DCs. The other rows indicate how the solution changes as more limited DCs are introduced into the problem. As can be seen from these tables, even though a large number of DCs have unlimited capacity, in most of the instances for the 15-node, 49-node, and 88-node data sets, the DCs with limited capacity are part of the solution. However, this is not the case for the 150-node data set as not only there are many more available nearby candidate DCs with unlimited capacity, but also the fixed cost for all the candidate DCs are the same. When the fixed cost is same across all DC locations, those uncapacitated candidate sites within a certain proximity of a capacitated DC site become more attractive candidates than that capacitated DC site.

As also can be seen from these four tables, the number of DCs are higher when the weight on transportation costs (β) are higher (i.e.,weight pair 1), as expected. However, the number of open DCs do not necessarily increase with an increase in the number of retailers because each retailer point is also a candidate DC location in these instances, which actually results in some cases in a decrease in the total cost. For example, the 49-node data set is a subset of the 88-node data set and the additional candidate DCs the 88-node data set has over the 49-node data set have lower fixed location costs. These additional candidate DCs allow for a potential decrease in transportation costs as well because there are more nearby candidate DCs available for those retailers common to both data sets.

Here, we also present a summary of the computational results for these 80 instances to facilitate the average and worst case performance of the developed heuristic for solving the CLMRP. Table 7 summarizes the average and maximum optimality gaps for problem instances of varying sizes and weights. The solution quality seems to be better when the relative weight on transportation costs to weight on inventory costs is higher (average gap of 0.014 vs. 0.205) or equivalently when the relative weight on the linear term to the weight on the nonlinear term in the objective function is higher. However, it should be noted that for larger problem instances, the average solution quality is comparable under both weight pairs.

Table 8 summarizes the average and worst-case CPU time requirements in seconds. The CPU time requirements are significantly smaller for weight pair 1 than weight pair 2. This difference is due to the CPU time requirements by the branch and bound method on the assignment variables, which are summarized in Table 9. The number of branch and bound iterations required for weight pair 1 is relatively few, because the number of open DCs in the optimal solution for weight pair 1 is a large portion of the total number of nodes in the network, that is, about 25% of the 88 candidate DC locations are open in the optimal solution. As a result, when we are using the Algorithm_C-SP$_j^{\mathrm{RP}}$ to solve the subproblem (**C-SP**$_j^{\mathbf{RP}}$), the set of candidate retailer locations for DC j, $|I^-|$ have a few elements. Therefore, the number of branch and bound iterations required to obtain the integral solution for DC j is more manageable.

The CPU time requirements although are significantly higher for weight pair 2, a worst case of about 212 min is not unreasonable for such a complex network design problem. However, as can be observed from the tables, the CPU times and the number of branch and bound iterations increase exponentially as the number of nodes in the network increases. One way to manage the CPU time requirements for larger data sets than the ones presented here is to not branch and bound on the assignment variables at earlier Lagrangian iterations. We experimented with this idea for the 88 and 150 node data sets for weight pair 2 and applied branch and bound for those Lagrangian iterations greater than 100 as opposed to 10. We observed a 85% reduction in CPU time requirements without necessarily a degrade in the solution quality (for more detailed results, see Ozsen (2004)).

6. CONCLUSIONS AND FUTURE RESEARCH

In this article, we have presented a new capacitated facility location model that incorporates risk pooling effects. This model generalizes the LMRP by adding capacities to the model in a novel way. This generalization is important for three reasons. First, the traditional capacitated fixed charge location models (CFLP) typically measure capacity in terms of the number of customers or demands per unit time that

Table 7. Average and worst-case solution quality.

	$\beta = 0.0004$ and $\theta = 0.01$		$\beta = 0.00001$ and $\theta = 0.001$	
	Avg gap	Max gap	Avg gap	Max gap
15 Node	0.000	0.000	0.346	1.099
49 Node	0.012	0.095	0.400	0.614
88 Node	0.040	0.211	0.069	0.599
150 Node	0.005	0.046	0.005	0.037
Overall	0.014	0.211	0.205	1.099

Table 8. Average and worst-case CPU times in seconds.

	$\beta = 0.0004$ and $\theta = 0.01$		$\beta = 0.00001$ and $\theta = 0.001$	
	Avg CPU	Max CPU	Avg CPU	Max CPU
15 Node	2	3	337	896
49 Node	76	219	1077	1630
88 Node	149	307	2684	12714
150 Node	131	310	2142	6896
Overall	89	310	1560	12714

Table 9. Number of branch and bound iterations on the assignment variables.

	$\beta = 0.0004$ and $\theta = 0.01$		$\beta = 0.00001$ and $\theta = 0.001$	
	Avg B&B iter	Max B&B iter	Avg B&B iter	Max B&B iter
15 Node	176	355	38,814	104,553
49 Node	8,188	24,483	126,781	194,414
88 Node	15,107	33,391	331,473	1,612,263
150 Node	12,250	31,742	261,641	849,127
Overall	8,930	33,391	189,677	1,612,263

can be served. Although this may be appropriate for some logistics systems, it remains meaningless for many others. In the CLMRP, warehouse capacity is defined in terms of the maximum inventory accumulation that is possible (or likely) during any cycle. Thus, it provides a far more reasonable measure of capacity for warehouses.

Second, by explicitly incorporating an inventory model into the location model, the CLMRP is able to add an additional dimension of choice when a company faces increased demand. In traditional capacitated facility location models, when demand increases there are only two options available: reallocate demands to more remote and more costly distribution centers, which are not at capacity or build additional facilities. The CLMRP allows for a third possibility, namely, that of ordering more often in smaller quantities, thereby reducing the inventory accumulated during any given cycle. In the short and intermediate terms, this is likely to be a far more cost-effective approach to managing demand increases.

Third, the CLMRP is structurally similar to the classical uncapacitated facility location model (UFLP). There is one extra term in the objective function of the CLMRP, which represents the optimal working inventory and safety stock costs. We assumed that the DCs use a (Q,r) model with type-1 service to replenish their inventories. We have proposed a Lagrangian-relaxation-based algorithm to solve this highly nonlinear, nonconvex integer-programming problem. The algorithm we have proposed can be used to solve for models in which the DCs replenish their inventory using different policies other than (Q,r) model. The only requirement is that the resulting optimal working inventory cost and safety stock cost function is differentiable.

In this formulation of the CLMRP, we considered the movement and storage of a single product. A natural extension to the CLMRP would be to consider multiple commodities and solve for assignment and order quantity variables simultaneously. Another natural extension to the CLMRP would be to allow the retailer to be sourced from multiple DCs to decrease the overall logistics costs. This natural extension is studied by Ozsen [25] and Ozsen et al. [26]. Also, in modeling the CLMRP, we have assumed direct shipments from the distribution centers to the assigned retailers. However, in practice, the shipments from a DC to the assigned retailers are often a traveling-salesman-like tour. Thus, we incorporate a better approximation of the shipment costs into the CLMRP (e.g., the approximations developed by Daganzo (1991) [6]).

Finally, the CLMRP does not consider congestion effects that might be experienced due to limited server availability at the loading/unloading docks but implicitly assumes that the bottleneck is the space availability. Sourirajan et al. [30] captures the trade-off between the risk-pooling effects and the congestion effects experienced by the DCs due to high volumes of demand being assigned to the DCs. However, their model does not solve for the shipment frequency between the plant and the DCs unlike the LMRP and the CLMRP. The CLMRP could be extended to capture these congestion effects. We are currently working on all of these extensions to the CLMRP.

APPENDIX A

In this appendix, we provide the proofs for the selected theorems.

PROOF OF THEOREM 1: We prove Property 1 using an interchange argument. Let $\mathbf{Y}^*_{\mathbf{ij}}$ be an optimal solution to the subproblem, ($\mathbf{C}$-$\mathbf{SP}^{\mathbf{RP}}_{\mathbf{j}}$) that does not satisfy Property 1 and contains two or more fractional values. Let $Y^*_{i_1j}$ and $Y^*_{i_2j}$ take on fractional values in the optimal solution. Then the objective function value of the solution $\mathbf{Y}^*_{\mathbf{ij}}$ can be computed as follows:

$$
\begin{aligned}
Z^*_j = & f_j + \sum_{i \in I^- \setminus \{i_1, i_2\}} A_i Y^*_{ij} + A_{i_1} Y^*_{i_1j} + A_{i_2} Y^*_{i_2j} \\
& + \bar{W}^*_j \left(\chi \left[\sum_{i \in I \setminus \{i_1, i_2\}} \mu_i Y^*_{ij} + \mu_{i_1} Y^*_{i_1j} + \mu_{i_2} Y^*_{i_2j} \right] \right)
\end{aligned}
$$

Now, let us define a new solution, $\mathbf{Y}'_{\mathbf{ij}}$ to the subproblem, ($\mathbf{C}$-$\mathbf{SP}^{\mathbf{RP}}_{\mathbf{j}}$), as follows:

$$
\mathbf{Y}'_{\mathbf{ij}} = \begin{cases} Y^*_{ij} & \forall i \in I \setminus \{i_1, i_2\} \\ Y'_{i_1j} & \text{if } i = i_1 \\ Y'_{i_2j} & \text{if } i = i_2 \end{cases}
$$

where $Y'_{i_1j} = Y^*_{i_1j} + \epsilon$ and $Y'_{i_2j} = Y^*_{i_2j} - \frac{\mu_{i_1}}{\mu_{i_2}}\epsilon$ for some $\epsilon > 0$, and i_1 is ranked before i_2.

For $\mathbf{Y}'_{\mathbf{ij}}$ to be a feasible solution to the subproblem, ($\mathbf{C}$-$\mathbf{SP}^{\mathbf{RP}}_{\mathbf{j}}$), we need to have

1. $Y'_{i_1j} = Y^*_{i_1j} + \epsilon \leq 1 \Rightarrow \epsilon \leq 1 - Y^*_{i_1j}$
2. $Y'_{i_2j} = Y^*_{i_2j} - \frac{\mu_{i_1}}{\mu_{i_2}}\epsilon \geq 0 \Rightarrow \epsilon \leq \frac{\mu_{i_2}}{\mu_{i_1}}(Y^*_{i_2j})$

Let us set $\epsilon = \min\{1 - Y^*_{i_1 j}, \frac{\mu_{i_2}}{\mu_{i_1}}(Y^*_{i_2 j})\}$. Then $\mathbf{Y}'_{\mathbf{ij}}$ is a feasible solution to the subproblem, (**C-SP$_\mathbf{j}^{\mathbf{RP}}$**). And the objective function value of the solution $\mathbf{Y}'_{\mathbf{ij}}$ is

$$\begin{aligned}
Z'_j =& f_j + \sum_{i\in I\setminus\{i_1,i_2\}} A_i Y^*_{ij} + A_{i_1}\left(Y^*_{i_1 j} + \epsilon\right) + A_{i_2}\left(Y^*_{i_2 j} - \frac{\mu_{i_1}}{\mu_{i_2}}\epsilon\right) \\
&+ \bar{W}^*_j\left(\chi\left[\sum_{i\in I\setminus\{i_1,i_2\}} \mu_i Y^*_{ij} + \mu_{i_1}\left(Y^*_{i_1 j} + \epsilon\right) + \mu_{i_2}\left(Y^*_{i_2 j} - \frac{\mu_{i_1}}{\mu_{i_2}}\epsilon\right)\right]\right) \\
=& f_j + \sum_{i\in I\setminus\{i_1,i_2\}} A_i Y^*_{ij} + A_{i_1} Y^*_{i_1 j} + A_{i_2} Y^*_{i_2 j} \\
&+ \epsilon\left(A_{i_1} - \frac{\mu_{i_1}}{\mu_{i_2}} A_{i_2}\right) + \bar{W}^*_j\left(\chi\left[\sum_{i\in I\setminus\{i_1,i_2\}} \mu_i Y^*_{ij}\right.\right. \\
&\left.\left. + \mu_{i_1} Y^*_{i_1 j} + \epsilon\mu_{i_1} + \mu_{i_2} Y^*_{i_2 j} - \epsilon\mu_{i_1}\right]\right)
\end{aligned}$$

Because i_1 is ranked before i_2, we have $\frac{A_{i_1}}{\mu_{i_1}} \le \frac{A_{i_2}}{\mu_{i_2}}$. Hence, $\epsilon(A_{i_1} - \frac{\mu_{i_1}}{\mu_{i_2}} A_{i_2}) \le 0$ and we have:

$$\begin{aligned}
Z'_j =& f_j + \sum_{i\in I\setminus\{i_1,i_2\}} A_i Y^*_{ij} + A_{i_1} Y^*_{i_1 j} + A_{i_2} Y^*_{i_2 j} + \epsilon\left(A_{i_1} - \frac{\mu_{i_1}}{\mu_{i_2}} A_{i_2}\right) \\
&+ \bar{W}^*_j\left(\chi\left[\sum_{i\in I\setminus\{i_1,i_2\}} \mu_i Y^*_{ij} + \mu_{i_1} Y^*_{i_1 j} + \mu_{i_2} Y^*_{i_2 j}\right]\right) \le Z^*_j
\end{aligned}$$

Hence, the solution $\mathbf{Y}^*_{\mathbf{ij}}$ is optimal. Now consider the following three possible values that ϵ can take on based on its definition:

- If $(1 - Y^*_{i_1 j}) < \frac{\mu_{i_2}}{\mu_{i_1}}(Y^*_{i_2 j})$, then $\epsilon = 1 - Y^*_{i_1 j}$ and $Y'_{i_1 j} = 1$ and $0 < Y'_{i_2 j} < 1$
- If $(1 - Y^*_{i_1 j}) > \frac{\mu_{i_2}}{\mu_{i_1}}(Y^*_{i_2 j})$, then $\epsilon = \frac{\mu_{i_2}}{\mu_{i_1}}(Y^*_{i_2 j})$ and $0 < Y'_{i_1 j} < 1$ and $Y'_{i_2 j} = 0$
- If $(1 - Y^*_{i_1 j}) = \frac{\mu_{i_2}}{\mu_{i_1}}(Y^*_{i_2 j})$, then $Y'_{i_1 j} = 1$ and $Y'_{i_2 j} = 0$

In each case, we have reduced the number of fractional variables by at least one without degrading the objective function value. Note that we may actually improve the objective function value, hereby disproving the assumed optimality of $\mathbf{Y}^*_{\mathbf{ij}}$. This argument can be repeated until at most one fractional value is included in the solution.

Property 2 follows from a similar interchange argument. □

PROOF OF THEOREM 2: The Lagrangian function for the problem (**C-SP$_\mathbf{j}^{\mathbf{RP}}$**) is

$$\begin{aligned}
L_j(Y,u,v) = f_j + \bar{W}^*_j\left(\chi\sum_{i\in I}\mu_i Y_{ij}\right) + \sum_{i\in I} A_i Y_{ij} \\
- \sum_{i\in I} u_i Y_{ij} + \sum_{i\in I} v_i (Y_{ij} - 1)
\end{aligned} \tag{18}$$

where we have introduced Lagrange multipliers u_i associated with the constraint $Y_{ij} \ge 0$ and v_i with the constraint $Y_{ij} \le 1$. Then the KKT conditions for the problem (**C-SP$_\mathbf{j}^{\mathbf{RP}}$**) are as follows:

$$\begin{aligned}
\frac{\partial L_j(Y,u,v)}{\partial Y_{ij}} &= \frac{\partial \bar{W}^*_j\left(\chi\sum_{i\in I}\mu_i Y_{ij}\right)}{\partial Y_{ij}} + A_i - u_i + v_i \\
&= 0, \forall i \in I & (19)\\
u_i Y_{ij} &= 0, \quad \forall i \in I & (20)\\
v_i(Y_{ij} - 1) &= 0, \quad \forall i \in I & (21)\\
0 \le Y_{ij} &\le 1, \quad \forall i \in I & (22)\\
u_i, v_i &\ge 0, \quad \forall i \in I & (23)
\end{aligned}$$

If $0 < Y^*_{kj} < 1$, then due to Eqs. (20) and (21) we have $u_k = v_k = 0$. Then from Eq. (19), we have

$$A_k = -\frac{\partial \bar{W}^*_j\left(\chi\sum_{i\in I}\mu_i Y^*_{ij}\right)}{\partial Y^*_{kj}}$$

as desired. □

PROOF OF THEOREM 3: Assume that we are processing node t of the branch and bound tree and that the constraint $Y_{kj} = 0$ is included in the current subproblem. Let $l > k$ having $A_l > A_k$ and $\mu_l > \mu_k$. We will show that it is valid to add the constraint $Y_{lj} = 0$ to node t and the nodes below node t. It suffices to show that no optimal solution at or below node t violates this constraint.

Let $\mathbf{Y}^*_{\mathbf{ij}}$ be an optimal solution to the IP (**C-SP$_\mathbf{j}$**). If $Y^*_{kj} = 1$, then we are finished, because this optimal solution is neither at or below node t. Assume $Y^*_{kj} = 0$. We will show $Y^*_{lj} = 0$ by contradiction. Suppose $Y^*_{lj} = 1$. Then the objective function value of the solution $\mathbf{Y}^*_{\mathbf{ij}}$ can be computed as follows:

$$\begin{aligned}
Z^*_j &= f_j + \sum_{i\in I^-\setminus\{k,l\}} A_i Y^*_{ij} + A_k Y^*_{kj} + A_l Y^*_{lj} \\
&\quad + \bar{W}^*_j\left(\chi\left[\sum_{i\in I\setminus\{k,l\}} \mu_i Y^*_{ij} + \mu_k Y^*_{kj} + \mu_l Y^*_{lj}\right]\right) \\
&= f_j + \sum_{i\in I\setminus\{k,l\}} A_i Y^*_{ij} + A_l + \bar{W}^*_j\left(\chi\left[\sum_{i\in I\setminus\{k,l\}} \mu_i Y^*_{ij} + \mu_l\right]\right)
\end{aligned}$$

Now, let us define a new solution, Y'_{ij} to the subproblem, (**C-SP$_\mathbf{j}$**), as follows:

$$Y'_{ij} = \begin{cases} Y^*_{ij} & \forall i \in I\setminus\{k,l\} \\ Y'_{kj} = 1 & \\ Y'_{lj} = 0 & \end{cases}$$

And the objective function value of the solution Y'_{ij} is

$$\begin{aligned}
Z'_j &= f_j + \sum_{i\in I\setminus\{k,l\}} A_i Y'_{ij} + A_k Y'_{kj} + A_l Y'_{lj} \\
&\quad + \bar{W}'_j\left(\chi\left[\sum_{i\in I\setminus\{k,l\}} \mu_i Y'_{ij} + \mu_k Y'_{kj} + \mu_l Y'_{lj}\right]\right)
\end{aligned}$$

$$= f_j + \sum_{i \in I \setminus \{k,l\}} A_i Y_{ij}^* + A_k + \bar{W}_j^* \left(\chi \left[\sum_{i \in I \setminus \{k,l\}} \mu_i Y_{ij}^* + \mu_k \right] \right)$$

$$\text{Then} Z_j' - Z_j^* = A_k - A_l + \bar{W}_j^* \left(\chi \left[\sum_{i \in I \setminus \{k,l\}} \mu_i Y_{ij}^* + \mu_k \right] \right)$$

$$- \bar{W}_j^* \left(\chi \left[\sum_{i \in I \setminus \{k,l\}} \mu_i Y_{ij}^* + \mu_l \right] \right)$$

Note that since $0 \geq A_l > A_k$ and $\mu_k < \mu_l$ and since $\bar{W}_j^*()$ is a monotone increasing function (see Appendix B), then $Z_j' - Z_j^* < 0$. Hence, Y_j^* is not optimal, which is a contradiction. □

APPENDIX B

Properties of $W_j(D_j)$ and $\bar{W}_j(D_j)$ Cost Functions

As introduced in Section 3:

$$\bar{W_j}^*(D_j) = W_j^*(D_j) + \theta h z_\alpha \sqrt{\frac{L_j D_j}{\chi}}$$

It can be shown that

$$W_j^*(D_j) = \begin{cases} W_{j,EOQ}^*(D_j) & 0 < D_j \leq \hat{D}_j \\ W_{j,C}^*(D_j) & \hat{D}_j \leq D_j < \bar{D}_j \end{cases}$$

where $W_{j,EOQ}^*(D_j) = \sqrt{2\theta h(F_j + \beta g_j)D_j} + \beta a_j D_j$

$$W_{j,C}^*(D_j) = \frac{(F_j + \beta g_j)D_j}{C_j - z_\alpha \sqrt{\frac{L_j D_j}{\chi}} - L_j \frac{D_j}{\chi}} + \beta a_j D_j + \frac{\theta h \left(C_j - z_\alpha \sqrt{\frac{L_j D_j}{\chi}} - L_j \frac{D_j}{\chi} \right)}{2}$$

$$\bar{D_j} = \frac{\chi}{2L_j} \left(2C - j + z_\alpha^2 - z_\alpha \sqrt{4C_j + z_\alpha^2} \right)$$

The following properties can be shown.

- $W_{j,EOQ}^*(D_j)$ is a concave function.
- $W_{j,C}^*(D_j)$ increases asymptotically as D_j approaches $\bar{D_j}$.
- $W_j^*(D_j)$ is a differentiable and a monotonically increasing function.
- $W_j{}^*(D_j)$ is neither a concave nor a convex function.
- $\bar{W_j}^*(D_j)$ is a differentiable and a monotonically increasing function.
- $\bar{W_j}^*(D_j)$ is neither a concave nor a convex function.

APPENDIX C

Capacity as a Chance Constraint

In this appendix, we argue that the capacity constraint constructed under the worst case scenario is not restrictive. Let ρ denote the probability that the inventory accumulation at DC exceeds the capacity of that DC. Let $D_{j,L}$ be the random variable denoting the demand at DC j during the lead time L_j. Then we have,

$$P(r_j - D_{j,L} + Q_j > C_j) \leq \rho$$
$$P(D_{j,L} < r_j + Q_j - C_j) \leq \rho$$

Because $D_{j,L}$ is approximately normally distributed with mean $L_j D_j / \chi$ and variance $L_j D_j / \chi$, we have

$$P\left(\frac{D_{j,L} - L_j D_j / \chi}{\sqrt{L_j D_j / \chi}} \leq \frac{r_j + Q_j - C_j - L_j D_j / \chi}{\sqrt{L_j D_j / \chi}} \right) \leq \rho$$

$$\frac{r_j + Q_j - C_j - L_j D_j / \chi}{\sqrt{L_j D_j / \chi}} \leq z_\rho$$

$$r_j + Q_j - C_j - L_j D_j / \chi \leq z_\rho \sqrt{L_j D_j / \chi}$$

$$Q_j + r_j - L_j D_j / \chi - z_\rho \sqrt{L_j D_j / \chi} \leq C_j$$

Since $r_j = z_\alpha \sqrt{L_j D_j / \chi} + L_j D_j / \chi$, we obtain

$$Q_j + z_\alpha \sqrt{L_j D_j / \chi} - z_\rho \sqrt{L_j D_j / \chi} \leq C_j$$
$$Q_j + (z_\alpha - z_\rho) \sqrt{L_j D_j / \chi} \leq C_j$$

Given this new constraint, the optimal order quantity will be as follows:

$$Q_j^* = \min \{ Q_{j,\text{EOQ}}, Q_{j,C}' \}.$$

where $Q_{j,C}' = C_j - (z_\alpha - z_\rho) \sqrt{L_j D_j / \chi}$.

It can be shown that the resulting optimal working inventory cost function is a differentiable and non decreasing function. Therefore, we can extend the models presented in this paper to use this sort of chance constraint and solve for the optimal solution using the solution algorithms described in section 4.1.

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